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Uniqueness for Fokker-Planck equations with measurable coefficients and applications to the fast diffusion equation.

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Abstract

The object of this paper is the uniqueness for a d-dimensional Fokker-Planck type equation with inhomogeneous (possibly degenerated) measurable not necessarily bounded coefficients. We provide an application to the probabilistic representation of the so-called Barenblatt's solution of the fast diffusion equation which is the partial differential equation $\partial_t u = \partial^2_{xx} u^m$ with $m \in]0,1[$. Together with the mentioned Fokker-Planck equation, we make use of small time density estimates uniformly with respect to the initial condition.

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1 Introduction

The present paper is divided into three parts.

i) A uniqueness result on a Fokker-Planck type equation with measurable non-negative (possibly degenerated) multidimensional unbounded coefficients.

ii) An application to the probabilistic representation of a fast diffusion equation.

iii) Some small time density estimates uniformly with respect to the initial condition.

In the whole paper $T > 0$ will stand for a fixed final time. In a one dimension space, the Fokker-Planck equation is of the type

$$\left\{ \begin{array}{rcl} \partial_t u(t,x) &=& \partial^2_{xx}(a(t,x)u(t,x)) - \partial_x(b(t,x)u(t,x)), \quad t \in]0,T], \ x \in \mathbb{R}, \\ u(0,\cdot) &=& \mu(dx), \end{array} \right. \tag{1.1}$$

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where $a, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable locally bounded coefficients and μ is a finite real Borel measure. The Fokker-Planck equation for measures is a widely studied subject in the literature whether in finite or infinite dimension. Recent work in the case of time-dependent coefficients with some minimal regularity was done by [9, 16, 30] in the case $d \geq 1$. In infinite dimension some interesting work was produced by [8].

In this paper we concentrate on the case of measurable (possibly) degenerate coefficients. Our interest is devoted to the irregularity of the diffusion coefficient, so we will set $b = 0$. A first result in that direction was produced in [7] where a was bounded, possibly degenerated, and the difference of two solutions was supposed to be in $L^2([\kappa, T] \times \mathbb{R})$, for every $\kappa > 0$ (ASSUMPTION (A)). This result was applied to study the probabilistic representation of a porous media type equation with irregular coefficients. We will later come back to this point. We remark that it is not possible to obtain uniqueness without ASSUMPTION (A). In particular [7, Remark 3.11] provides two measure-valued solutions when a is time-homogeneous, continuous, with $\frac{1}{a}$ integrable in a neighborhood of zero.

One natural question is about what happens when a is not bounded and $x \in \mathbb{R}^d$. A partial answer to this question is given in Theorem 3.1 which is probably the most important result of the paper; it is a generalization of [7, Theorem 3.8] where the inhomogeneous function a was bounded. Theorem 3.1 handles the multidimensional case and it allows a to be unbounded.

An application of Theorem 3.1 concerns the parabolic problem:

$$\begin{cases} \partial_t u(t,x) &= \partial^2_{xx}(u^m(t,x)), \quad t \in]0,T], \ x \in \mathbb{R}, \\ u(0,\cdot) &= \delta_0, \end{cases} \tag{1.2}$$

where δ_0 is the Dirac measure at zero and u^m denotes $u|u|^{m-1}$. It is well known that, for $m > 1$, there exists an exact solution to (1.2), the so-called *Barenblatt's density*, see [3]. Its explicit formula is recalled for instance in [34, Chapter 4] and more precisely in [4, Section 6.1]. Equation (1.2) is the *classical* porous medium equation.

In this paper, we focus on (1.2) when $m \in]0, 1[$: the *fast diffusion equation*. In fact, an analogous Barenblatt type solution also exists in this case, see [34, Chapter 4] and references therein; it is given by the expression

$$\mathcal{U}(t,x) = t^{-\alpha}\left(D + \tilde{k}|x|^2 t^{-2\alpha}\right)^{-\frac{1}{1-m}}, \tag{1.3}$$

where

$$\alpha = \frac{1}{m+1}, \ \tilde{k} = \frac{1-m}{2(m+1)m}, \ D = \left(\frac{I}{\sqrt{\tilde{k}}}\right)^{\frac{2(1-m)}{m+1}}, \ I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(x)]^{\frac{2m}{1-m}}\, dx. \tag{1.4}$$

Equation (1.2) is a particular case of the so-called generalized porous media type equation

$$\begin{cases} \partial_t u(t,x) &= \partial^2_{xx}\beta\left(u(t,x)\right), \quad t \in]0,T], \\ u(0,x) &= u_0(dx), \quad x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a monotone non-decreasing function such that $\beta(0) = 0$ and u_0 is a finite measure. When $\beta(u) = u^m, \ m \in]0,1[$ and $u_0 = \delta_0$, two difficulties arise: first, the coefficient β is of singular type since it is not locally Lipschitz, second, the initial condition is a measure. Another type of singular coefficient is $\beta(u) = H(u - u_c)u$, where H is a Heaviside function and $u_c > 0$ is some critical value, see e.g. [2]. Problem (1.2) with $m \in]0,1[$ was studied by several authors. For a bounded integrable function as initial condition, the equation in (1.2) is well-stated in the sense of distributions, as a by product of the classical papers [10, 6] on (1.5) with general monotonous coefficient β.

When the initial data is locally integrable, existence was proved by [19]. [11] extended the validity of this result when u_0 is a finite Radon measure in a bounded domain, [29] established existence when u_0 is a locally finite measure in the whole space. The Barenblatt's solution is an *extended continuous solution* as defined in [13, 14]; [14, Theorem 5.2] showed uniqueness in that class. [23, Theorem 3.6] showed existence in a bounded domain of solutions to the fast diffusion equation perturbed by a right-hand side source term, being a general finite and positive Borel measure. As far as we know, there is no uniqueness argument in the literature whenever the initial condition is a finite measure in the general sense of distributions. Among recent contributions, [15] investigated the large time behavior of solutions to (1.2).

The present paper provides the probabilistic representation of the (Barenblatt's) solution of (1.2) and exploits this fact in order to approach it via a Monte Carlo simulation with an L^2 error around 10^{-3}. We make use of the probabilistic procedure developed in [4, Section 4] and we compare it to the exact form of the solution $\mathcal{U}$ of (1.2) which is given by the explicit formulae (1.3)-(1.4). The target of [4] was the case $\beta(u) = H(u - u_c)u$; in that paper those techniques were compared with a deterministic numerical analysis recently developed in [12] which was very performing in that target case. At this stage, the implementation of the same deterministic method for the fast diffusion equation does not give satisfying results; this constitutes a further justification for the probabilistic representation.

We define
$$\Phi(u) = |u|^{\frac{m-1}{2}}, \quad u \in \mathbb{R}, \quad m \in]0,1[.$$
The probabilistic representation of $\mathcal{U}$ consists in finding a suitable stochastic process Y such that the law of Y_t has $\mathcal{U}(t,\cdot)$ as density. Y will be a (weak) solution of the non-linear SDE
$$\begin{cases} Y_t &= \int_0^t \sqrt{2}\Phi(\mathcal{U}(s,Y_s))dW_s, \\ \mathcal{U}(t,\cdot) &= \text{Law density of } Y_t, \quad \forall\, t \in]0,T], \end{cases} \tag{1.6}$$
where W is a Brownian motion on some suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

To the best of our knowledge, the first author who considered a probabilistic representation of a solution of (1.5) was H. P. Jr. McKean ([26]), particularly in relation with the so-called *propagation of chaos*. In his case β was smooth, but the equation also included a first order coefficient. From then on, literature steadily grew and nowadays there is a vast amount of contributions to the subject, especially when the nonlinearity is in the first order part, as e.g. in Burgers' equation. We refer the reader to the excellent survey papers [33] and [18]. A probabilistic interpretation of (1.5) when $\beta(u) = u.|u|^{m-1},\ m > 1$, was provided for instance in [5]. Recent developments related to chaos propagation when $\beta(u) = u^2$ and $\beta(u) = u^m, m > 1$ were proposed in [28] and [17]. The probabilistic representation in the case of possibly discontinuous β was treated in [7] when β is non-degenerate and in [2] when β is degenerate; the latter case includes the case $\beta(u) = H(u - u_c)u$.

As a preamble to the probabilistic representation we make a simple, yet crucial observation. Let W be a standard Brownian motion.

Proposition 1.1. *Let $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(u) = \Phi^2(u).u$, $\Phi : \mathbb{R} \to \mathbb{R}_+$ and u_0 be a probability real measure.*

Let Y be a solution to the problem
$$\begin{cases} Y_t &= Y_0 + \int_0^t \sqrt{2}\Phi(u(s,Y_s))dW_s, \\ u(t,\cdot) &= \text{Law density of } Y_t, \quad \forall\, t \in]0,T], \\ u(0,\cdot) &= u_0(dx). \end{cases} \tag{1.7}$$

Then $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is solution to* (1.5)*.*

Proof of the above result is based on the following lemma.

Lemma 1.2. *Let* $a : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ *be measurable. Let* (Y_t) *be a process which solves the SDE*

$$Y_t = Y_0 + \int_0^t \sqrt{2a(s,Y_s)}dW_s, \quad t \in [0,T].$$

Consider the function $t \mapsto \rho(t,\cdot)$ *from* $[0,T]$ *to the space of finite real measures* $\mathcal{M}(\mathbb{R})$*, defined as* $\rho(t,\cdot)$ *being the law of* Y_t*. Then* ρ *is a solution, in the sense of distributions (see* (2.2)*), of*

$$\begin{cases} \partial_t u &= \partial^2_{xx}(au), \ t \in]0,T], \\ u(0,\cdot) &= \textit{Law of } Y_0. \end{cases} \tag{1.8}$$

Proof of Lemma 1.2. This is a classical result, see for instance [32, Chapter 4]. The proof is based on an application of Itô's formula to $\varphi(Y_t)$, $\varphi \in \mathcal{S}(\mathbb{R})$. □

Proof of Proposition 1.1. We set $a(s,y) = \Phi^2(u(s,y))$. We apply Lemma 1.2 setting $\rho(t,dy) = u(t,y)dy$, $t \in]0,T]$, and $\rho(0,\cdot) = u_0$. □

When u_0 is the Dirac measure at zero and $\beta(u) = u^m$, with $m \in]\frac{3}{5},1[$, Theorem 5.7 states the converse of Proposition 1.1, providing a process Y which is the unique (weak) solution of (1.6). The first step consists in reducing the proof of that Theorem to the proof of Proposition 5.3 where the Dirac measure, as initial condition of (1.2), is replaced by the function $\mathcal{U}(\kappa,\cdot)$, $0 < \kappa \le T$. This corresponds to the shifted Barenblatt's solution along a time κ, which will be denoted by $\overline{\mathcal{U}}$. Also, in this case Proposition 5.3 provides an unique strong solution of the corresponding non-linear SDE. That reduction is possible through a weak convergence argument of the solutions given by Proposition 5.3 when $\kappa \to 0$. The idea of the proof of Proposition 5.3 is the following. Let W be a standard Brownian motion and $\overline{Y_0}$ be a r.v. distributed as $\mathcal{U}(\kappa,\cdot)$; since $\Phi(\overline{\mathcal{U}})$ is Lipschitz, the SDE

$$\overline{Y_t} = \overline{Y_0} + \int_0^t \Phi(\overline{\mathcal{U}}(s,\overline{Y_s}))dW_s, \ t \in]0,T],$$

admits a unique strong solution. The marginal laws of $(\overline{Y_t})$ and $\overline{\mathcal{U}}$ can be shown to be both solutions to (1.8) for $a(s,y) = (\overline{\mathcal{U}}(s,y))^{m-1}$; that a will be denoted in the sequel by $\bar{a}$. The leading argument of the proof is carried by Theorem 3.1 which states uniqueness for measure valued solutions of the Fokker-Planck type PDE (1.8) under some **Hypothesis(B)**. More precisely, to conclude that the marginal laws of $(\overline{Y_t})$ and $\overline{\mathcal{U}}$ coincide via Theorem 3.1, we show that they both verify the so-called **Hypothesis(B2)**. In order to prove that for $\overline{\mathcal{U}}$, we will make use of Lemma 4.2. The verification of **Hypothesis(B2)** for the marginal laws of $\overline{Y}$ is more involved. It makes use of a small time (uniformly with respect to the initial condition) upper bound for the density of an inhomogeneous diffusion flow with linear growth (unbounded) smooth coefficients, even though the diffusion term is non-degenerate and all the derivatives are bounded. This is the object of Proposition 5.1, the proof of which is based on an application of Malliavin calculus. In our opinion this result alone is of interest as we were not able to find it in the literature. When the paper was practically finished we discovered an interesting recent result of M. Pierre, presented in [20, Chapter 6], obtained independently. This result holds in dimension 1 when the coefficients are locally bounded, non-degenerate and the initial condition has a first moment. In this case, the hypothesis of type (B) is not needed. In particular it allows one to establish Proposition 5.3, but not Theorem 5.7 where the coefficients are not locally bounded on $[0,T] \times \mathbb{R}$.

The paper is organized as follows. Section 2 is devoted to basic notations. Section 3 is concentrated on Theorem 3.1 which concerns uniqueness for the deterministic, time inhomogeneous, Fokker-Planck type equation. Section 4 presents some properties of the Barenblatt's solution $\mathcal{U}$ to (1.2). The probabilistic representation of $\mathcal{U}$ is treated in Section 5. Proposition 5.1 performs small time density estimates for time-inhomogeneous diffusions, the proof of which is located in the Appendix. Finally, Section 6 is devoted to numerical experiments.

2 Preliminaries

We start with some basic analytical framework. In the whole paper d will be a strictly positive integer. If $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded function we will denote $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$. By $S(\mathbb{R}^d)$ we denote the space of rapidly decreasing infinitely differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, by $S'(\mathbb{R}^d)$ its dual (the space of tempered distributions). We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of finite Borel measures on $\mathbb{R}^d$. If $x \in \mathbb{R}^d$, $|x|$ will denote the usual Euclidean norm.

For $\varepsilon > 0$, let K_ε be the Green's function of $\varepsilon - \Delta$, that is the kernel of the operator $(\varepsilon - \Delta)^{-1} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. In particular, for all $\varphi \in L^2(\mathbb{R}^d)$, we have

$$B_\varepsilon \varphi := (\varepsilon - \Delta)^{-1}\varphi(x) = \int_{\mathbb{R}} K_\varepsilon\,(x-y)\,\varphi(y)dy. \tag{2.1}$$

For more information about the corresponding analysis, the reader can consult [31]. If $\varphi \in C^2(\mathbb{R}^d) \bigcap S'(\mathbb{R}^d)$, then $(\varepsilon - \Delta)\varphi$ coincides with the classical associated PDE operator evaluated at φ.

Definition 2.1. *We will say that a function $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is non-degenerate if there is a constant $c_0 > 0$ such that $\psi \geq c_0$.*

Definition 2.2. *We will say that a function $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ has linear growth (with respect to the second variable) if there is a constant C such that $|\psi(\cdot, x)| \leq C(1 + |x|)$, $x \in \mathbb{R}$.*

Definition 2.3. *Let $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}_+$ be a Borel function, $z^0 \in \mathcal{M}(\mathbb{R}^d)$. A (weakly measurable) function $z : [0,T] \to \mathcal{M}(\mathbb{R}^d)$ is said to be a solution in the sense of distributions of*

$$\partial_t z = \Delta(az)$$

with initial condition $z(0,\cdot) = z^0$ if, for every $t \in [0,T]$ and $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}^d} \phi(x) z(t,dx) = \int_{\mathbb{R}^d} \phi(x) z^0(dx) + \int_0^t ds \int_{\mathbb{R}^d} \Delta\phi(x) a(s,x) z(s,dx). \tag{2.2}$$

3 Uniqueness for the Fokker-Planck equation

We now state the main result of the paper which concerns uniqueness for the Fokker-Planck type equation with measurable, time-dependent, (possibly degenerated and unbounded) coefficients. It generalizes [7, Theorem 3.8] where the coefficients were bounded and one-dimensional.

The theorem below holds with two classes of hypotheses: **(B1)**, operating in the multidimensional case, and **(B2)**, more specifically in the one-dimensional case.

Theorem 3.1. *Let a be a Borel nonnegative function on $[0,T] \times \mathbb{R}^d$. Let $z_i : [0,T] \to \mathcal{M}(\mathbb{R}^d)$, $i = 1,2$, be continuous with respect to the weak topology on finite measures on $\mathcal{M}(\mathbb{R}^d)$. Let z^0 be an element of $\mathcal{M}(\mathbb{R}^d)$. Suppose that both z_1 and z_2 solve the problem $\partial_t z = \Delta(az)$ in the sense of distributions with initial condition $z(0,\cdot) = z^0$.*

Then $z := (z_1 - z_2)(t,\cdot)$ is identically zero for every t under the following requirement.

Hypothesis (B). There is $\tilde z \in L^1_{loc}([0,T]\times\mathbb{R}^d)$ such that $z(t,\cdot)$ admits $\tilde z(t,\cdot)$ as density for almost all $t\in[0,T]$; $\tilde z$ will still be denoted by z. Moreover, either **(B1)** or **(B2)** below is fulfilled.

$$\textbf{(B1)}\quad \text{(i)} \int\limits_{[0,T]\times\mathbb{R}^d} |z(t,x)|^2\,dt\,dx < +\infty, \quad \text{(ii)} \int\limits_{[0,T]\times\mathbb{R}^d} |az|^2(t,x)dtdx < +\infty.$$

(B2) We suppose $d=1$. For every $t_0>0$, we have

$$\text{(i)} \int\limits_{[t_0,T]\times\mathbb{R}} |z(t,x)|^2\,dt\,dx < +\infty, \quad \text{(ii)} \int\limits_{[0,T]\times\mathbb{R}} |az|(t,x)\,dt\,dx < +\infty, \quad \text{(iii)} \int\limits_{[t_0,T]\times\mathbb{R}} |az|^2(t,x)\,dt\,dx < +\infty.$$

Remark 3.2. The weak continuity of $z(t,\cdot)$ and [7, Remark 3.10] imply that $\sup\limits_{t\in[0,T]} \|z(t,\cdot)\|_{var} < +\infty$, where $\|\cdot\|_{var}$ denotes the total variation. In particular $\sup\limits_{0<t\le T} \int_{\mathbb{R}^d} |z(t,x)|dx < +\infty$.

Remark 3.3. 1. If a is bounded then the first item of Hypothesis(B1) implies the second one.
2. If a is non-degenerated, assumption (ii) of Hypothesis(B1) implies assumption (i).

Remark 3.4. Let $d=1$.

1. If a is non-degenerate, the third assumption of Hypothesis(B2) implies the first one.
2. If $z(t,x)\in L^\infty([t_0,T]\times\mathbb{R})$ then the first item of Hypothesis(B2) is always verified.
3. If a is bounded then assumption (ii) of Hypothesis(B2) is always verified by Remark 3.2; the first item of Hypothesis(B2) implies the third one. So Theorem 3.1 is a strict generalization of [7, Theorem 3.8].
4. Let $(z(t,\cdot), t\in[0,T])$ be the marginal law densities of a stochastic process Y solving
$$Y_t = Y_0 + \int_0^t \sqrt{2a(s,Y_s)}dW_s,$$
with Y_0 distributed as z^0 such that $\int\limits_{\mathbb{R}} |x|^2 z^0(dx) < +\infty$.

   If $\sqrt{a}$ has linear growth, it is well known that $\sup\limits_{t\le T}\mathbb{E}(|Y_t|^2) < +\infty$; so
$$\int\limits_{[0,T]\times\mathbb{R}} |a(s,x)z(s,x)|\,ds\,dx = \mathbb{E}\left[\int\limits_0^T a(s,Y_s)\,ds\right] < +\infty.$$
   Therefore assumption (ii) in Hypothesis(B2) is always fulfilled.

Proof of Theorem 3.1. Let z_1, z_2 be two solutions of (2.2); we set $z := z_1 - z_2$. We evaluate, for every $t\in[0,T]$, the quantity
$$g_\varepsilon(t) = \|z(t,\cdot)\|^2_{-1,\varepsilon},$$
where $\|f\|_{-1,\varepsilon} = \|(\varepsilon-\Delta)^{-\frac12} f\|_{L^2}$.

Similarly to the first part of the proof of [7, Theorem 3.8], assuming we can show that
$$\lim_{\varepsilon\to 0} g_\varepsilon(t) = 0, \quad \forall t\in[0,T], \tag{3.1}$$
we are able to prove that $z(t)\equiv 0$ for all $t\in]0,T]$. We explain this fact.

Let $t \in]0,T]$. We recall the notation $B_\varepsilon f = (\varepsilon - \Delta)^{-1} f$, if $f \in L^2(\mathbb{R}^d)$. Since $z(t,\cdot) \in L^2(\mathbb{R}^d)$ then $B_\varepsilon z(t,\cdot) \in H^2(\mathbb{R}^d)$ and so $\nabla B_\varepsilon z(t,\cdot) \in H^1(\mathbb{R}^d)^d \subset L^2(\mathbb{R}^d)^d$. This gives

$$\begin{aligned} g_\varepsilon(t) &= \int_{\mathbb{R}^d} B_\varepsilon z(t,x) z(t,x) dx = \varepsilon \int_{\mathbb{R}^d} (B_\varepsilon z(t,x))^2 dx - \int_{\mathbb{R}^d} B_\varepsilon z(t,x) \Delta B_\varepsilon z(t,x) dx \\ &= \varepsilon \int_{\mathbb{R}^d} (B_\varepsilon z(t,x))^2 dx + \int_{\mathbb{R}^d} |\nabla B_\varepsilon z(t,x)|^2 dx. \end{aligned}$$

Since the two terms of the above sum are non-negative, if (3.1) holds, then $\sqrt{\varepsilon} B_\varepsilon z(t,\cdot) \to 0$ (resp. $|\nabla B_\varepsilon z(t,\cdot)| \to 0$) in $L^2(\mathbb{R}^d)$ (resp. in $L^2(\mathbb{R}^d)^d$). So, for all $t \in]0,T]$, $z(t,\cdot) = \varepsilon B_\varepsilon z(t,\cdot) - \Delta B_\varepsilon z(t,\cdot) \to 0$, in the sense of distributions, as ε goes to zero. Therefore $z \equiv 0$.

We proceed now with the proof of (3.1). We have the following identities in the sense of distributions:

$$z(t,\cdot) = \int_0^t \Delta(az)(s,\cdot) ds = \int_0^t (\Delta - \varepsilon)(az)(s,\cdot) ds + \varepsilon \int_0^t (az)(s,\cdot) ds, \tag{3.2}$$

which implies

$$B_\varepsilon z(t,\cdot) \quad = \quad - \int_0^t (az)(s,\cdot) ds + \varepsilon \int_0^t B_\varepsilon (az)(s,\cdot) ds. \tag{3.3}$$

Let $\delta > 0$ and (ϕ_δ) a sequence of mollifiers converging to the Dirac delta function at zero. We set $z_\delta(t,x) = \int_{\mathbb{R}^d} z(t,y) \phi_\delta(x-y) dy$, observing that $z_\delta \in (L^1 \bigcap L^\infty)([0,T] \times \mathbb{R}^d)$. Moreover, (3.2) gives

$$z_\delta(t,\cdot) = \int_0^t \Delta(az)_\delta(s,\cdot) ds.$$

We suppose now Hypothesis(B1) (resp. (B2)). Let $t_0 = 0$ (resp. $t_0 > 0$). By assumption (B1)(ii) (resp. (B2)(iii)), we have $\Delta(az)_\delta \in L^2([t_0,T] \times \mathbb{R}^d)$. Thus, z_δ can be seen as a function belonging to $C([t_0,T]; L^2(\mathbb{R}^d))$. Besides, identities (3.2) and (3.3) lead to

$$\begin{aligned} z_\delta(t,\cdot) &= z_\delta(t_0,\cdot) + \int_{t_0}^t (\Delta - \varepsilon)(az)_\delta(s,\cdot) ds + \varepsilon \int_{t_0}^t (az)_\delta(s,\cdot) ds, && (3.4) \\ B_\varepsilon z_\delta(t,\cdot) &= B_\varepsilon z_\delta(t_0,\cdot) - \int_{t_0}^t (az)_\delta(s,\cdot) ds + \varepsilon \int_{t_0}^t B_\varepsilon (az)_\delta(s,\cdot) ds. && (3.5) \end{aligned}$$

Proceeding through integration by parts with values in $L^2(\mathbb{R}^d)$, we get

$$\begin{aligned} \|z_\delta(t,\cdot)\|^2_{-1,\varepsilon} - \|z_\delta(t_0,\cdot)\|^2_{-1,\varepsilon} &= -2 \int_{t_0}^t ds < z_\delta(s,\cdot), (az)_\delta(s,\cdot) >_{L^2} \\ &\quad + 2\varepsilon \int_{t_0}^t ds < (az)_\delta(s,\cdot), B_\varepsilon z_\delta(s,\cdot) >_{L^2} . \end{aligned} \tag{3.6}$$

Then, letting δ go to zero, using assumptions (B1)(i)-(ii) (resp. (B2)(i) and (B2)(iii)) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|z(t,\cdot)\|^2_{-1,\varepsilon} - \|z(t_0,\cdot)\|^2_{-1,\varepsilon} &= -2 \int_{t_0}^t ds \int_{\mathbb{R}^d} a(s,x) |z|^2(s,x) dx \\ &\quad + 2\varepsilon \int_{t_0}^t ds < (az)(s,\cdot), B_\varepsilon z(s,\cdot) >_{L^2} . \end{aligned} \tag{3.7}$$

At this stage of the proof, we assume that **Hypothesis(B1)** is satisfied. Since $t_0 = 0$, we have $z(t_0, \cdot) = 0$. Using the inequality $c_1 c_2 \le \frac{c_1^2 + c_2^2}{2}$, c_1, $c_2 \in \mathbb{R}$ and Cauchy-Schwarz, (3.7) implies

$$\begin{aligned} \|z(t,\cdot)\|^2_{-1,\varepsilon} &\le -2\int_0^t ds \int_{\mathbb{R}^d} (a|z|^2)(s,x)dx + \varepsilon \int_0^t ds \|az(s,\cdot)\|^2_{L^2} + \varepsilon \int_0^t ds \|B_\varepsilon z(s,\cdot)\|^2_{L^2} \\ &\le \varepsilon \int_0^t ds \|az(s,\cdot)\|^2_{L^2} + \int_0^t ds \|z(s,\cdot)\|^2_{-1,\varepsilon}, \end{aligned} \tag{3.8}$$

because for $f = z(s, \cdot)$, we have

$$\varepsilon \|B_\varepsilon f\|^2_{L^2} = \varepsilon \int_{\mathbb{R}^d} \frac{(\mathcal{F}(f))^2(\xi)}{(\varepsilon + |\xi|^2)^2} d\xi \le \int_{\mathbb{R}^d} \frac{(\mathcal{F}(f))^2(\xi)}{\varepsilon + |\xi|^2} d\xi = \|f\|^2_{-1,\varepsilon}.$$

We observe that the first integral of the right-hand side of (3.8) is finite by assumption (B1)(ii). Gronwall's lemma, applied to (3.8), gives

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} \le \varepsilon e^T \int_0^T ds \|az(s,\cdot)\|^2_{L^2}.$$

Letting $\varepsilon \to 0$, it follows that $\|z(t,\cdot)\|^2_{-1,\varepsilon} = 0, \ \forall t \in [0,T]$. This concludes the first part of the proof.

We now suppose that **Hypothesis(B2)** is satisfied, in particular $d = 1$. By [7, Lemma 2.2] we have

$$\sup_x 2\varepsilon |B_\varepsilon z(s,x)| \le \sqrt{\varepsilon} \|z(s,\cdot)\|_{var}.$$

Consequently (3.7) gives

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} - \|z(t_0,\cdot)\|^2_{-1,\varepsilon} \le \sqrt{\varepsilon} \sup_{t \le T} \|z(t,\cdot)\|_{var} \int\limits_{[t_0,T]\times\mathbb{R}} |az|(s,x) ds dx. \tag{3.9}$$

Besides, arguing like in the proof of [7, Theorem 3.8], we obtain that

$$\lim_{t_0 \to 0} \|z(t_0,\cdot)\|^2_{-1,\varepsilon} = 0.$$

We first let $t_0 \to 0$ in (3.9), which implies

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} \le \sqrt{\varepsilon} \sup_{t \le T} \|z(t,\cdot)\|_{var} \int\limits_{[0,T]\times\mathbb{R}} |az|(s,x) ds; \tag{3.10}$$

we remark that the right-hand side of (3.10) is finite by assumption (B2)(ii). Letting ε go to zero, the proof of (3.1) is finally established.

□

4 Basic facts on the fast diffusion equation

We go on providing some properties of the Barenblatt's solution $\mathcal{U}$ to (1.2) when $m \in]0,1[$ and given by (1.3)-(1.4).

Proposition 4.1.

(i) $\mathcal{U}$ *is a solution in the sense of distributions to* (1.2)*. In particular, for every* $\varphi \in C_0^\infty(\mathbb{R})$*, we have*

$$\int\limits_{\mathbb{R}} \varphi(x)\mathcal{U}(t,x)dx = \varphi(0) + \int\limits_0^t ds \int\limits_{\mathbb{R}} \mathcal{U}^m(s,x)\varphi''(x)dx. \tag{4.1}$$

(ii) $\int\limits_{\mathbb{R}} \mathcal{U}(t,x)dx = 1, \quad \forall t > 0$. *In particular, for any* $t > 0$*,* $\mathcal{U}(t,\cdot)$ *is a probability density.*

(iii) The Dirac measure δ_0 *is the initial trace of* $\mathcal{U}$*, in the sense that*

$$\int\limits_{\mathbb{R}} \gamma(x)\mathcal{U}(t,x)dx \to \gamma(0), \text{ as } t \to 0, \tag{4.2}$$

for every $\gamma : \mathbb{R} \to \mathbb{R}$*, continuous and bounded.*

Proof of Proposition 4.1. (i) This is a well known fact which can be established by inspection.

(ii) For $M \geq 1$, we consider a sequence of smooth functions (φ^M), such that

$$\varphi^M(x) \begin{cases} = 0, & \text{if } |x| \geq M+1; \\ \leq 1, & \text{if } |x| \in [M, M+1]; \\ = 1, & \text{if } |x| \leq M. \end{cases}$$

By (4.1) we have

$$\int\limits_{\mathbb{R}} \varphi^M(x)\mathcal{U}(t,x)dx = 1 + \int\limits_0^t ds \int\limits_{\mathbb{R}} \mathcal{U}^m(s,x)(\varphi^M)''(x)dx. \tag{4.3}$$

Letting $M \to +\infty$, by Lebesgue's dominated convergence theorem, the left-hand side of (4.3) converges to $\int_{\mathbb{R}} \mathcal{U}(t,x)dx$. The integral on the right-hand side of (4.3) is bounded by

$$C\int\limits_0^t ds \int\limits_M^{M+1} \mathcal{U}^m(s,x)dx \leq C\int\limits_0^t s^{-\alpha m}\left(D + \tilde{k}M^2 s^{-2\alpha}\right)^{\frac{-m}{1-m}} ds \leq \frac{C}{(\tilde{k}M^2)^{\frac{m}{1-m}}} \int\limits_0^T s^{\frac{m}{1-m}} ds.$$

The last integral on the right is finite as $\frac{m}{1-m} > 0$, for every $m \in]0,1[$. Therefore the integral in the right-hand side of (4.3) goes to zero as $M \to +\infty$. This concludes the proof of the second item of Proposition 4.1.

(iii) (4.2) follows by elementary changes of variables. □

Note that the second item of Proposition 4.1 determines the explicit expression of the constant D.

Lemma 4.2.

(i) Suppose that $\frac{1}{3} < m < 1$ *. Then there is* $p \geq 2$ *and a constant* $\mathcal{C}_p$ *(depending on* T*) such that for* $0 \leq s < \ell \leq T$

$$\int\limits_{]s,\ell]\times\mathbb{R}} dtdx\,(\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1} \leq \mathcal{C}_p(\ell - s). \tag{4.4}$$

(ii) In particular, taking $p = 2$ *in* (4.4)*, we get*

$$\int\limits_{]0,T]\times\mathbb{R}} dtdx\,(\mathcal{U}(t,x))^m < +\infty, \tag{4.5}$$

again when m *belongs to* $]\frac{1}{3}, 1[$*.*

(iii) If $\frac{1}{5} < m < 1$ *,*

$$\int\limits_{]0,T]\times\mathbb{R}} dtdx\,(\mathcal{U}(t,x))^{2m} < +\infty. \tag{4.6}$$

(iv) If m *belongs to* $]\frac{3}{5},1[$ *, then*

$$\forall\kappa > 0,\ \int\limits_{\mathbb{R}} |x|^4\mathcal{U}(\kappa,x)dx < +\infty. \tag{4.7}$$

Proof of Lemma 4.2.

(i) Using (1.3), we have

$$\int\limits_{]s,\ell]\times\mathbb{R}} (\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1}dtdx = \int\limits_{]s,\ell]\times\mathbb{R}} t^{\frac{-\alpha p(m-1)}{2}-\alpha}\Big(D+\tilde{k}|x|^2t^{-2\alpha}\Big)^{\frac{p}{2}-\frac{1}{1-m}}dtdx.$$

Then, setting $y = t^{-\alpha}x\sqrt{\frac{\tilde{k}}{D}}$, we get

$$\begin{aligned}\int\limits_{]s,\ell]\times\mathbb{R}} (\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1}dtdx &= \frac{D^{\frac{p+1}{2}-\frac{1}{1-m}}}{\sqrt{\tilde{k}}}\int\limits_{s}^{l} t^{\frac{p}{2}\alpha(1-m)}dt\int_{\mathbb{R}}(1+y^2)^{\frac{p}{2}-\frac{1}{1-m}}dy \\ &\le \frac{D^{\frac{p+1}{2}-\frac{1}{1-m}}}{\sqrt{\tilde{k}}}T^{\frac{p}{2}\alpha(1-m)}(\ell-s)\int_{\mathbb{R}}(1+y^2)^{\frac{p}{2}-\frac{1}{1-m}}dy.\end{aligned}$$

The last integral is finite if $(p+1)(1-m) < 2$. This implies (4.4).

(ii) is a particular case of (i) and (iii) follows by similar arguments as for the proof of (i).

(iv) Now we assume that $m \in]\frac{3}{5},1[$. For $\kappa > 0$ we have

$$\int\limits_{\mathbb{R}} |x|^4\mathcal{U}(\kappa,x)dx = \frac{D^{\frac{3-5m}{2(1-m)}}}{\tilde{k}^{5/2}}\kappa^{4\alpha}\int\limits_{\mathbb{R}}|y|^4(1+y^2)^{-\frac{1}{1-m}}dy, \tag{4.8}$$

where this last equality was obtained setting $y = \kappa^{-\alpha}x\sqrt{\frac{\tilde{k}}{D}}$. Clearly, since $m \in]\frac{3}{5},1[$, the integral in the right-hand side of (4.8) is finite. Therefore (4.7) is fulfilled. □

Let $\kappa \in]0,T]$. Given $u : [0,T]\times\mathbb{R} \to \mathbb{R}$ we associate

$$\overline{u}(t,x) = u(t+\kappa,x), \quad (t,x) \in [0,T-\kappa]\times\mathbb{R}. \tag{4.9}$$

In particular we have

$$\overline{\mathcal{U}}(t,x) = \mathcal{U}(t+\kappa,x). \tag{4.10}$$

Moreover, for every $x \in \mathbb{R}$, we denote

$$u_{0,\kappa}(x) = \mathcal{U}(\kappa,x). \tag{4.11}$$

Remark 4.3. Function $\overline{\mathcal{U}}$ solves the problem

$$\left\{\begin{array}{rcl}\partial_t u &=& \partial_{xx}^2(u^m), \\ u(0,\cdot) &=& u_{0,\kappa}\ .\end{array}\right. \tag{4.12}$$

5 The probabilistic representation of the fast diffusion equation

We are now interested in a non-linear stochastic differential equation rendering the probabilistic representation related to (1.2) and given by (1.6). Suppose for a moment that Y_0 is a random variable distributed according to δ_0, so $Y_0 = 0$ a.s. We recall that, if there exists a process Y being a solution in law of (1.6), then Proposition 1.1 implies that u solves (1.2) in the sense of distributions.

In this subsection we shall prove existence and uniqueness of solutions in law for (1.6). In this respect we first state a tool, given by Proposition 5.1 below, concerning the existence of an upper bound for the marginal law densities of the solution Y of an inhomogeneous SDE with unbounded coefficients. This result has an independent interest.

Proposition 5.1. *Let $\sigma,\ b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous (not necessarily bounded) functions such that $\sigma(t,\cdot)$, $b(t,\cdot)$ are smooth with bounded derivatives of orders greater or equal than one. σ is supposed to be non-degenerate.*

Let $x_0 \in \mathbb{R}$ and $Y_t = (Y_t^{x_0})_{t\in[0,T]}$ be the solution of

$$Y_t = x_0 + \int_0^t \sigma(r, Y_r) dW_r + \int_0^t b(r, Y_r) dr. \tag{5.1}$$

Then, for every $s > 0$, the law of Y_s admits a density denoted $p_s(x_0, \cdot)$.

Moreover, we have

$$p_s(x_0, x) \le \frac{K}{\sqrt{s}} \left(1 + |x_0|^4\right),\ \forall (s,x) \in]0,T] \times \mathbb{R}, \tag{5.2}$$

where K is a constant which depends on $\|\sigma'\|_\infty$, $\|b'\|_\infty$ and T but not on x_0.

Remark 5.2.

1. The proof of Proposition 5.1 above is given in Appendix 7.1.
2. If σ and b is bounded, the classical Aronson's estimates implies that (5.2) holds even without the $|x_0|^4$ multiplicative term. If σ and b are unbounded, [1] provides an adaptation of Aronson's estimates; unfortunately they first considered time-homogeneous coefficients, and also their result does not imply (5.2).
3. If σ and b have polynomial growth and are time-homogeneous, various estimates are given in [25]. However the behavior is of type $\mathcal{O}(t^{-\frac{3}{2}})$ instead of $\mathcal{O}(t^{-\frac{1}{2}})$ when $t \to 0$.

Let Y_κ be a random variable distributed according to $u_{0,\kappa}$. We are interested in the following result.

Proposition 5.3. *Assume that $m \in]\frac{3}{5}, 1[$. Let B be a classical Brownian motion independent of Y_κ. Then there exists a unique (strong) solution $\overline{Y} = (\overline{Y_t})_{t\in[0,T-\kappa]}$ of*

$$\begin{cases} \overline{Y_t} &= Y_\kappa + \int\limits_0^t \Phi(\overline{\mathcal{U}}(s, \overline{Y_s})) dB_s, \\ \overline{\mathcal{U}}(t,\cdot) &= \text{Law density of } \overline{Y_t}, \quad \forall\, t \in [0, T-\kappa], \\ \overline{\mathcal{U}}(0,\cdot) &= u_{0,\kappa}\,. \end{cases} \tag{5.3}$$

In particular pathwise uniqueness holds.

Corollary 5.4. *Let W be a classical Brownian motion independent of Y_κ. Therefore there is a unique (strong) solution $Y^\kappa = (Y_t^\kappa)_{t\in[\kappa,T]}$ of*

$$\begin{cases} Y_t^\kappa &= Y_\kappa + \int\limits_\kappa^t \Phi(\mathcal{U}(s, Y_s^\kappa)) dW_s, \\ \mathcal{U}(t,\cdot) &= \text{Law density of } Y_t^\kappa, \quad \forall\, t \in [\kappa, T], \\ \mathcal{U}(\kappa,\cdot) &= u_{0,\kappa}\,. \end{cases} \tag{5.4}$$

Proof of Corollary 5.4. We start with the proof of uniqueness. Let $\kappa > 0$. We consider two solutions $Y^{\kappa,1}$ and $Y^{\kappa,2}$ of (5.4), we set $\overline{Y_t^i} = Y^{\kappa,i}_{t+\kappa}$, $\forall t \in [0, T-\kappa]$, $i = 1, 2$ and $B_t = W_{t+k} - W_t$, $\forall t \in [0, T-\kappa]$. Clearly $\overline{Y_t^1}$ and $\overline{Y_t^2}$ solve (5.3). Therefore, using Proposition 5.3, we deduce uniqueness for problem (5.4). Existence follows by similar arguments. □

Proof of Proposition 5.3. Let W be a classical Brownian motion on some filtered probability space. Given the function $\overline{\mathcal{U}}$, defined in (4.10), we construct below a unique process $\overline{Y}$ strong solution of

$$\overline{Y_t} = \overline{Y_0} + \int_0^t \Phi(\overline{\mathcal{U}}(s, \overline{Y_s}))dW_s. \tag{5.5}$$

From (4.10), for every $(s, y) \in [0, T-\kappa] \times \mathbb{R}$, we have

$$\Phi(\overline{\mathcal{U}}(s, y)) = \sqrt{2\bar{a}(s, y)},$$

where

$$\bar{a}(s, y) = (s+\kappa)^{\alpha(1-m)}(D + \tilde{k}|y|^2(s+\kappa)^{-2\alpha}). \tag{5.6}$$

In fact, $\Phi(\overline{\mathcal{U}})$ is continuous, smooth with respect to the space parameter and all the space derivatives of order greater or equal than one are bounded; in particular $\Phi(\overline{\mathcal{U}})$ is Lipschitz and it has linear growth. Therefore (5.5) admits a strong solution.

By Lemma 1.2 the function $t \mapsto \rho(t, \cdot)$ from $[0, T-\kappa]$ to $\mathcal{M}(\mathbb{R})$, where $\rho(t, \cdot)$ is the law of $\overline{Y_t}$, is a solution to

$$\begin{cases} \partial_t \rho &= \partial^2_{xx}(\bar{a}\rho), \\ \rho(0, \cdot) &= u_{0,\kappa}. \end{cases} \tag{5.7}$$

To conclude it remains to prove that $\overline{\mathcal{U}}(t, y)dy$ is the law of $\overline{Y_t}$, $\forall t \in [0, T-\kappa]$; in particular the law of the r.v. $\overline{Y_t}$ admits a density. For this we will apply Theorem 3.1 for which we need to check the validity of Hypothesis(B2) when $a = \bar{a}$ and for $z := z_1 - z_2$, where $z_1 := \rho$ and $z_2 := \overline{\mathcal{U}}$. By additivity this will be of course fulfilled if we prove it separately for $z := \rho$ and $z := \overline{\mathcal{U}}$, which are both solutions to (5.7).

Since $\bar{a}$ is non-degenerate, by Remark 3.4(1), we only need to check items (ii) and (iii) of the mentioned Hypothesis(B2). On one hand, since $\bar{a}(s, y) = \overline{\mathcal{U}}^{m-1}(s, y)$, $z := \overline{\mathcal{U}}$ verifies Hypothesis(B2) because of items (ii) and (iii) of Lemma 4.2. On the other hand, since $\sqrt{\bar{a}}$ has linear growth, by Remark 3.4.(4) ρ fulfills item (ii) of Hypothesis(B2). Moreover, by Lemma 5.5 below, ρ also verifies item (iii) of Hypothesis(B2). Finally Theorem 3.1 implies that $\overline{\mathcal{U}} \equiv \rho$.

□

Lemma 5.5. *Let $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}_+$, continuous (not necessarily bounded) such that $\psi(t, \cdot)$ is smooth with bounded derivatives of orders greater or equal than one. We also suppose ψ to be non-degenerate.*

We consider a stochastic process $X = (X_t)_{t\in[0,T]}$ strong solution of the SDE

$$X_t = X_0 + \int_0^t \psi(s, X_s)dW_s, \tag{5.8}$$

where X_0 is a random variable distributed according to $u_{0,\kappa}$ defined in (4.11) with $m \in]\frac{3}{5}, 1[$.

For $t \in]0, T]$ the law of X_t has a density $\nu(t, \cdot)$ such that $(\psi^2\nu)(t, x)$ belongs to $L^2([t_0, T] \times \mathbb{R})$, for every $t_0 > 0$.

Proof of Lemma 5.5.

If $X_0 = x_0$, where x_0 is a real number, then Proposition 5.1 implies that, for every $t \in]0,T]$, the law of X_t admits a density $p_t(x_0,\cdot)$. Consequently, if the law of X_0 is $u_{0,\kappa}(x)dx$, for every $t \in]0,T]$, the law of X_t has a density given by

$$\nu(t,x) = \int_{\mathbb{R}} u_{0,\kappa}(x_0) p_t(x_0,x) dx_0.$$

By (5.2) in Proposition 5.1 it follows

$$\sup_{(t,x)\in[t_0,T]\times\mathbb{R}} p_t(x_0,x) \le K_0(1+|x_0|^4), \quad \text{where} \quad K_0 = \frac{K}{\sqrt{t_0}}. \tag{5.9}$$

Using (5.9) we get

$$K_1 := \sup_{(t,x)\in[t_0,T]\times\mathbb{R}} |\nu(t,x)| \le K_0 \int_{\mathbb{R}} (1+|x_0|^4)\mathcal{U}(\kappa,x_0)dx_0 < \infty; \tag{5.10}$$

the latter inequality is valid because of (4.7) in Lemma 4.2. In the sequel of the proof, the constants K_2, K_3, K_4 will only depend on t_0, T and ψ. Furthermore

$$\int_{[t_0,T]\times\mathbb{R}} \left((\psi^2\nu)(t,x)\right)^2 dtdx \le \sup_{(t,x)\in[t_0,T]\times\mathbb{R}} |\nu(t,x)| \mathbb{E}\left[\int_0^T \psi^4(t,X_t)dt\right].$$

Since ψ has linear growth, this expression is bounded by

$$K_1K_2\left(1+\int_0^T \mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right]dt\right). \tag{5.11}$$

(5.11) follows because of (5.10). Besides, by Burkholder-Davis-Gundy and Jensen's inequalities, taking into account the linear growth of ψ, it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] \le K_3\left(\mathbb{E}\left[|X_0|^4\right] + \int_0^T \mathbb{E}\left[\sup_{s\in[0,T]}|X_s|^4\right]ds + T\right).$$

Then, by Gronwall's lemma, there is another constant K_4 such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] \le K_4\left(1+\int_{\mathbb{R}}|x_0|^4\mathcal{U}(\kappa,x_0)dx_0\right). \tag{5.12}$$

Finally (5.11), (5.12) and (5.10) allow us to conclude the proof. □

We are now ready to provide the probabilistic representation related to function $\mathcal{U}$ which in fact is only a solution in law of (1.6).

Definition 5.6. *We say that* (1.6) *admits a weak (in law) solution if there is a probability space* $(\Omega,\mathcal{F},\mathbb{P})$*, a Brownian motion* $(W_t)_{t\ge0}$ *and a process* $(Y_t)_{t\ge0}$ *such that the system* (1.6) *holds.* (1.6) *admits uniqueness in law if, given* (W^1,Y^1)*,* (W^2,Y^2) *solving* (1.6) *on some related probability space, it follows that* Y^1 *and* Y^2 *have the same law.*

Theorem 5.7. *Assume that* $m \in]\frac{3}{5},1[$*. Then there is a unique weak solution (in law)* Y *of problem* (1.6).

Remark 5.8. Indeed the assumption on $m \in]\frac{3}{5}, 1[$ is only required for the application of Theorem 3.1. The arguments following the present proof only use $m > \frac{1}{3}$.

Proof of Theorem 5.7. First we start with the existence of a weak solution for (1.6). Let $\mathcal{U}$ be again the (Barenblatt's) solution of (1.2). We consider the solution $(Y_t^\kappa)_{t\in[\kappa,T]}$ provided by Corollary 5.4 extended to $[0,\kappa]$, setting $Y_t^\kappa = Y_\kappa$, $t \in [0,\kappa]$. We prove that the laws of processes Y^κ are tight. For this we implement the classical Kolmogorov's criterion, see [22, Section 2.4, Problem 4.11]. We will show the existence of $p > 2$ such that

$$\mathbb{E}\left[|Y_t^\kappa - Y_s^\kappa|^p\right] \le \mathcal{C}_p |t-s|^{\frac{p}{2}}, \quad \forall s,t \in [0,T], \tag{5.13}$$

where $\mathcal{C}_p$ will stand for a constant (not always the same), depending on p and T but not on κ. Let $s,t \in]0,T]$. Let $p > 2$. By Burkholder-Davis-Gundy inequality we obtain

$$\mathbb{E}\left[|Y_t^\kappa - Y_s^\kappa|^p\right] \le \mathcal{C}_p \mathbb{E}\left[\left|\int_s^t \Phi^2(\mathcal{U}(r, Y_r^\kappa))dr\right|^{\frac{p}{2}}\right].$$

Then, using Jensen's inequality and the fact that $\mathcal{U}(r,\cdot)$ is the law density of Y_r^κ, $r \ge \kappa$, we get

$$\mathbb{E}\left[|Y_t^\kappa - Y_s^\kappa|^p\right] \le \mathcal{C}_p |t-s|^{\frac{p}{2}-1} \int_s^t dr \int_{\mathbb{R}} \Phi^p(\mathcal{U}(r,y))\mathcal{U}(r,y)dy. \tag{5.14}$$

We have

$$\int_s^t dr \int_{\mathbb{R}} \Phi^p(\mathcal{U}(r,y))\mathcal{U}(r,y)dy = \int_s^t dr \int_{\mathbb{R}} dy \, (\mathcal{U}(r,y))^{\frac{p(m-1)}{2}+1},$$

and, by Lemma 4.2 (i), the result follows.

Consequently there is a subsequence $Y^n := Y^{\kappa_n}$ converging in law (as $C([0,T])-$valued random elements) to some process Y. Let P^n be the corresponding laws on the canonical space $\Omega = C([0,T])$ equipped with the Borel σ-field. Y will denote the canonical process $Y_t(\omega) = \omega(t)$. Let P be the weak limit of (P^n).

1) We first observe that the marginal laws of Y under P^n converge to the marginal law of Y under P. Let $t \in]0,T]$. If the sequence (κ_n) is lower than t, then the law of Y_t under P^n equals the constant law $\mathcal{U}(t,x)dx$. Consequently, for every $t \in]0,T]$, the law of Y_t under P is $\mathcal{U}(t,x)dx$.

2) We now prove that Y is a (weak) solution of (1.6), under P. By similar arguments as for the classical stochastic differential equations, see [32, Chapter 6], it is enough to prove that Y (under P) fulfills the martingale problem i.e., for every $f \in C_b^2(\mathbb{R})$, the process

$$\textbf{(MP)} \quad f(Y_t) - f(0) - \frac{1}{2}\int_0^t f''(Y_s)\Phi^2(\mathcal{U}(s,Y_s))ds,$$

is an $(\mathcal{F}_s)$-martingale, where $(\mathcal{F}_s)$ is the canonical filtration associated with Y. $C_b^2(\mathbb{R})$ stands for the set $\{f \in C^2(\mathbb{R}) | f, f', f'' \text{ bounded }\}$. Let $\mathbb{E}$ (resp. $\mathbb{E}^n$) be the expectation operator with respect to P (resp. P^n). Let $s,t \in [0,T]$ with $s < t$ and $R = R(Y_r, r \le s)$ be an $\mathcal{F}_s-$measurable, bounded and continuous (on $C([0,T])$) random variable. In order to show the martingale property **(MP)** of Y, we have to prove that

$$\mathbb{E}\left[\left(f(Y_t) - f(Y_s) - \frac{1}{2}\int_s^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right) R\right] = 0, \quad f \in C_b^2(\mathbb{R}). \tag{5.15}$$

We first consider the case when $s > 0$. There is $n \geq n_0$, such that $\kappa_n < s$. Let $f \in C_b^2(\mathbb{R})$; since $(Y_s)_{s \geq \kappa_n}$, under P^n, are still martingales we have

$$\mathbb{E}^n \left[\left(f(Y_t) - f(Y_s) - \frac{1}{2} \int_s^t f''(Y_r) \Phi^2(\mathcal{U}(r, Y_r)) dr \right) R \right] = 0. \tag{5.16}$$

We are able to prove that (5.15) follows from (5.16). Let $\varepsilon > 0$ and $N > 0$ such that

$$\int_s^t dr \int_{\{|y| > \frac{N}{C} - 1\}} \mathcal{U}^m(r, y) dy \leq \varepsilon, \tag{5.17}$$

where C is the linear growth constant of $\Phi^2 \circ \mathcal{U}$ in the sense of Definition 2.2. In order to conclude, passing to the limit in (5.16), we will only have to show that

$$\lim_{n \to +\infty} \mathbb{E}^n \left[F(Y) \right] - \mathbb{E} \left[F(Y) \right] = 0, \tag{5.18}$$

where $F(\ell) = \int_s^t dr \Phi^2(\mathcal{U}(r, \ell(r)) f''(\ell(r)) R(\ell(\xi), \xi \leq s)$, $F : C([0, T]) \to \mathbb{R}$ being continuous but not bounded. The left-hand side of (5.18) equals

$$\begin{aligned} &\mathbb{E}^n \left[F(Y) - F^N(Y) \right] + \mathbb{E}^n \left[F^N(Y) \right] - \mathbb{E} \left[F^N(Y) \right] + \mathbb{E} \left[F^N(Y) - F(Y) \right] \\ &:= \mathcal{E}_1(n, N) + \mathcal{E}_2(n, N) + \mathcal{E}_3(n, N), \end{aligned} \tag{5.19}$$

where

$$F^N(\ell) = \int_s^t dr \left(\Phi^2(\mathcal{U}(r, \ell(r)) \wedge N \right) f''(\ell(r)) R(\ell(\xi), \xi \leq s).$$

Since $\kappa_n < s$, for N large enough, we get

$$\begin{aligned} |\mathcal{E}_1(n, N)| &\leq \|R\|_\infty \|f''\|_\infty \int_s^t dr \int_{\{\Phi^2(\mathcal{U}(r,y)) \geq N\}} \left(\Phi^2(\mathcal{U}(r, y) - N \right) \mathcal{U}(r, y) dy \\ &\leq \|R\|_\infty \|f''\|_\infty \int_s^t dr \int_{\{|y| > \frac{N}{C} - 1\}} \mathcal{U}^m(r, y) dy \leq \varepsilon \|R\|_\infty \|f''\|_\infty, \end{aligned} \tag{5.20}$$

taking into account (5.17) and the second item of Lemma 4.2. For fixed N, chosen in (5.17), we have $\lim_{n \to +\infty} \mathcal{E}_2(n, N) = 0$, since F^N is bounded and continuous. Again, since the law density under P of Y_t, $t \geq s$, is $\mathcal{U}(t, \cdot)$, similarly as for (5.20), we obtain $|\mathcal{E}_3(n, N)| \leq \varepsilon \|R\|_\infty \|f''\|_\infty$. Finally, coming back to (5.19), it follows

$$\limsup_{n \to +\infty} \left| \mathbb{E}^n \left[F(Y) \right] - \mathbb{E} \left[F(Y) \right] \right| \leq 2\varepsilon \|R\|_\infty \|f''\|_\infty;$$

since $\varepsilon > 0$ is arbitrary, (5.18) is established. So (5.15) is verified for $s > 0$.

3) Now, we consider the case when $s = 0$. We first prove that

$$\mathbb{E} \left[\int_0^T \Phi^2(\mathcal{U}(r, Y_r)) dr \right] < +\infty. \tag{5.21}$$

By item 1) of this proof, the law of Y_r, $r > 0$ admits $\mathcal{U}(r,\cdot)$ as density. Consequently, the left-hand side of (5.21) gives

$$\int\limits_{]0,T]} dr \int\limits_{\mathbb{R}} \Phi^2(\mathcal{U}(r,y))\mathcal{U}(r,y)dy = \int\limits_0^T dr \int\limits_{\mathbb{R}} \mathcal{U}^m(r,y)dy,$$

which is finite by the second item of Lemma 4.2. Coming back to (5.15), we can now let s go to zero. Since Y is continuous and f is bounded, we clearly have $\lim_{s\to 0} \mathbb{E}\left[f(Y_s)R\right] = \mathbb{E}\left[f(Y_0)R\right]$. Moreover

$$\lim_{s\to 0} \mathbb{E}\left[\left(\int\limits_s^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right) R\right] = \mathbb{E}\left[\left(\int\limits_0^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right) R\right],$$

using Lebesgue's dominated convergence theorem and (5.21). Consequently we obtain

$$\mathbb{E}\left[\left(f(Y_t) - f(Y_0) - \frac{1}{2}\int\limits_0^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right) R\right] = 0. \tag{5.22}$$

It remains to show that $Y_0 = 0$ a.s. This follows because $Y_t \to Y_0$ a.s., and also in law (to δ_0), by the third item of Proposition 4.1. Finally we have shown that the limiting process Y verifies **(MP)**, which proves the existence of solutions to (1.6).

4) We now prove uniqueness. Since $\mathcal{U}$ is fixed, only uniqueness for the first line of equation (1.6) has to be established. Let $(Y^i_t)_{t\in[0,T]}$, $i = 1, 2$, be two solutions. In order to show that the laws of Y^1 and Y^2 are identical, according to [21, Lemma 2.5], we will verify that their finite marginal distributions are the same. For this we consider $0 = t_0 < t_1 < \ldots < t_N = T$. Let $0 < \kappa < t_1$. Obviously we have $Y^i_{t_0} = 0$ a.s., in the corresponding probability space, $\forall i \in \{1,2\}$. Both restrictions $Y^1|_{[\kappa,T]}$ and $Y^2|_{[\kappa,T]}$ verify (5.4). Since that equation admits pathwise uniqueness, it also admits uniqueness in law by Yamada-Watanabe theorem. Consequently $Y^1|_{[\kappa,T]}$ and $Y^2|_{[\kappa,T]}$ have the same law and in conclusion the law of $(Y^1_{t_1},\ldots,Y^1_{t_N})$ coincides with the law of $(Y^2_{t_1},\ldots,Y^2_{t_N})$, thus, the law of $(Y^1_{t_0},\ldots,Y^1_{t_N})$ coincides with the law of $(Y^2_{t_0},\ldots,Y^2_{t_N})$. □

6 Numerical experiments

In order to avoid singularity problems due to the initial condition being a Dirac delta function, we will consider a time translation of $\mathcal{U}$, denoted v, and defined by

$$v(t,\cdot) = \mathcal{U}(t+1,\cdot), \quad \forall t \in [0,T].$$

v still solves equation (1.2), for $m \in]0,1[$, but with now a smooth initial data given by

$$v_0(x) = \mathcal{U}(1,x), \quad \forall x \in \mathbb{R}. \tag{6.1}$$

Indeed, we have the following formula

$$v(t,x) = (t+1)^{-\alpha}\left(D + \tilde{k}|x|^2(t+1)^{-2\alpha}\right)^{-\frac{1}{1-m}}, \tag{6.2}$$

where α, $\tilde{k}$ and D are still given by (1.4).

We now wish to compare the exact solution of problem (1.2) to a numerical probabilistic solution. In fact, in order to perform such approximated solutions, we use the algorithm described in Sections 4 of [4] (implemented in Matlab). We focus on the case

$m = \frac{1}{2}$.

Simulation experiments: we compute the numerical solution over the time-space grid $[0, 1.5] \times [-15, 15]$. We use $n = 50000$ particles and a time step $\Delta t = 2 \times 10^{-4}$. Figures 1.(a)-(b)-(c)-(d), display the exact and the numerical solutions at times $t = 0$, $t = 0.5$, $t = 1$ and $t = T = 1.5$, respectively. The exact solution for the fast diffusion equation (1.2), given in (6.2), is depicted by solid lines. Besides, Figure 1.(e) describes the time evolution of the discrete L^2 error on the time interval $[0, 1.5]$.

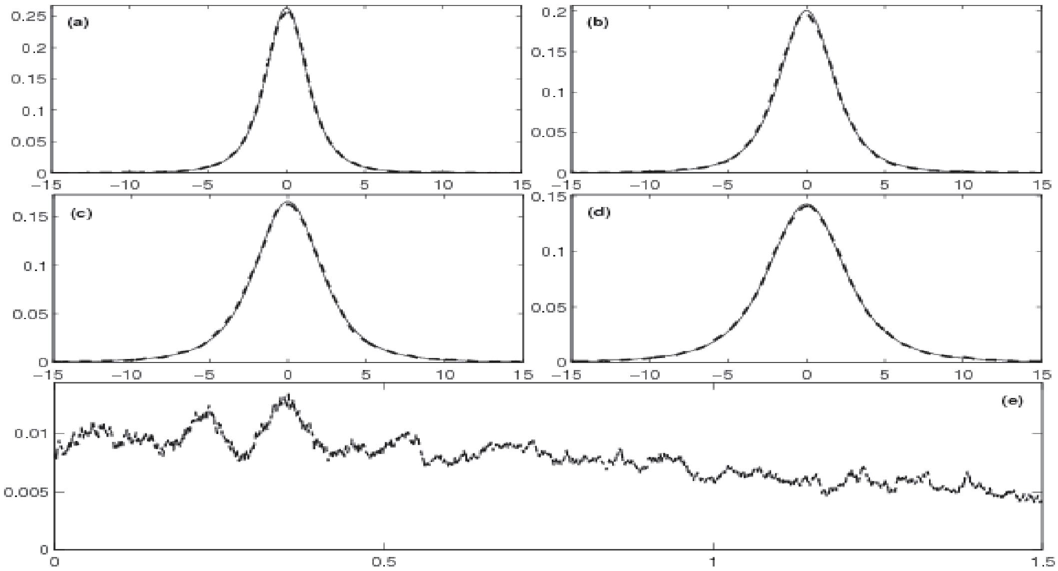

Figure 1: Numerical (dashed line) and exact solutions (solid line) values at t=0 (a), t=0.5 (b), t=1 (c) and t=1.5 (d). The evolution of the L^2 error over the time interval $[0, 1.5]$ (e).

7 Appendix

7.1 Proof of Proposition 5.1

We start with some notations for the Malliavin calculus. The set $\mathbb{D}^\infty$ represents the classical Sobolev-Malliavin space of smooth test random variables. $\mathbb{D}^{1,2}$ is defined in the lines after [27, Lemma 1.2.2] and $\mathbb{L}^{1,2}$ is introduced in [27, Definition 1.3.2]. See also [24] for a complete monograph on Malliavin calculus. We state a preliminary result.

Proposition 7.1. *Let N be a non-negative random variable. Suppose, for every $p \geq 1$, the existence of constants $C(p)$ and $\epsilon_0(p)$ such that*

$$P(N \leq \epsilon) \leq C(p).\epsilon^{p+1}, \quad \forall \epsilon \in]0, \epsilon_0(p)]. \tag{7.1}$$

Then, for every $p \geq 1$,

$$\mathbb{E}(N^{-p}) \leq \epsilon_0(p).C(p+1) + \epsilon_0(p)^{-p} P(N > \epsilon_0(p)). \tag{7.2}$$

Proof of Proposition 7.1. Let $p \geq 1$ and $\epsilon_0(p) > 0$. Setting $F(x) = P(N \leq x)$, $x \in \mathbb{R}_+$, we have

$$\mathbb{E}(N^{-p}) = I_1 + I_2, \tag{7.3}$$

where

$$I_1 = \int_0^{\epsilon_0(p)} x^{-p} dF(x) \text{ and } I_2 = \int_{\epsilon_0(p)}^{+\infty} x^{-p} dF(x).$$

(7.1) implies that I_1 and I_2 are well-defined. Indeed, on one hand, applying integration by parts on I_1, we get

$$I_1 = \left[x^{-p}F(x)\right]_0^{\epsilon_0(p)} + p \int_0^{\epsilon_0(p)} x^{-p-1}F(x)dx;$$

moreover, there is a constant $C(p)$ such that

$$I_1 \le (p+1)\epsilon_0(p)C(p). \tag{7.4}$$

On the other hand, again (7.1) says that

$$I_2 \le \epsilon_0(p)^{-p}(1 - F(\epsilon_0(p))). \tag{7.5}$$

Consequently, using (7.4) and (7.5) and coming back to (7.3), (7.2) is established. □

Proof of Proposition 5.1. In this proof σ' (resp. b') stands for $\partial_x\sigma$ (resp. $\partial_x b$). Let $Y = (Y_t^{x_0})_{t\in[0,T]}$, be the solution of (5.1). According to [27, Theorem 2.2.2] we have $Y_s \in \mathbb{D}^\infty$, $\forall s \in [0,T]$. Let $s > 0$. Since σ is non-degenerate, by [27, Theorem 2.3.1], the law of Y_s admits a density that we denote by $p_s(x_0, \cdot)$.

The second step consists in a re-scaling, transforming the time s into a noise multiplicative parameter λ; we set $\lambda = \sqrt{s}$. Indeed, (Y_t) is distributed as $(Y^\lambda_{\frac{t}{\lambda^2}})$, where

$$Y_t^\lambda = x_0 + \lambda \int_0^t \sigma(r\lambda^2, Y_r^\lambda)dW_r + \lambda^2 \int_0^t b(r\lambda^2, Y_r^\lambda)dr.$$

In particular, $Y_s \sim Y_1^\lambda$. By previous arguments, for every $t > 0$, $Y_t^\lambda \in \mathbb{D}^\infty$ and its law admits a density denoted by $p_t^\lambda(x_0, \cdot)$. Our aim consists in showing the existence of a constant K such that

$$p_1^\lambda(x_0, y) \le \frac{K}{\lambda}(1 + |x_0|^4),\ \forall y \in \mathbb{R}, \lambda \in]0, \sqrt{T}], \tag{7.6}$$

where K is a constant which does not depend on x_0 and λ. In fact, we will prove that, for every $\lambda \in]0, \sqrt{T}]$,

$$\sup_{y\in\mathbb{R}, t\in]0,1]} p_t^\lambda(x_0, y) \le \frac{K}{\lambda}(1 + |x_0|^4). \tag{7.7}$$

We set $Z_t^\lambda = \dfrac{Y_t^\lambda - x_0}{\lambda}$, $\ t \in [0,1]$, so that the density q_t^λ of Z_t^λ fulfills $q_t^\lambda(z) = \lambda p_t^\lambda(x_0, \lambda z + x_0)$, $(t,z) \in [0,1] \times \mathbb{R}$. In fact, we will have attained (7.7), if we show

$$\sup_{z\in\mathbb{R}, \lambda\in]0,\sqrt{T}]} q_t^\lambda(z) \le K(1 + |x_0|^4),\ t \in]0,1]. \tag{7.8}$$

We express the equation fulfilled by Z; it yields

$$Z_t^\lambda = \int_0^t \sigma^\lambda(r, Z_r^\lambda)dW_r + \int_0^t b^\lambda(r, Z_r^\lambda)dr, \tag{7.9}$$

where, for every $(r,z) \in [0,1] \times \mathbb{R}$, we set

$$\sigma^\lambda(r,z) = \sigma(r\lambda^2, \lambda z + x_0), \quad \text{and } b^\lambda(r,z) = \lambda b(r\lambda^2, \lambda z + x_0).$$

At this stage we state the following lemma.

Lemma 7.2. *For $\lambda \in]0,1]$, we shorten by $Z := (Z_t^\lambda)_{t\in[0,T]}$, the solution of* (7.9). *For every $\gamma \geq 1$, we have*

$$\sup_{\lambda\in]0,\sqrt{T}]} \mathbb{E}\left[\sup_{t\in[0,1]} |Z_t|^\gamma\right] \leq C(1+|x_0|^\gamma), \tag{7.10}$$

where C is a constant depending on $\|\sigma'\|_\infty$, $\|b'\|_\infty$ and T, but not on x_0.

Remark 7.3.

1. For simplicity, in the whole proof of Proposition 5.1, we will set $T = 1$.
2. Since there is no more ambiguity, we will use again the letter s in the considered integrals.

Proof of Lemma 7.2. Let $\lambda \in]0,1]$ and $\gamma \geq 1$. In the proof C_1 is a constant depending on T, and C_2, C_3 depend on T, $\|\sigma'\|_\infty$ and $\|b'\|_\infty$. Using Burkholder-Davis-Gundy and Jensen's inequalities, we get

$$\mathbb{E}\left[\sup_{\rho\in[0,t]} |Z_\rho|^\gamma\right] \leq C_1 \left(\int_0^t \mathbb{E}\left[|\sigma(s\lambda^2, \lambda Z_s + x_0)|^\gamma\right] ds + \int_0^t \mathbb{E}\left[|\lambda b(s\lambda^2, \lambda Z_s + x_0)|^\gamma\right] ds\right).$$

Since σ' and b' are bounded, σ and b have linear growth. Therefore, previous expression is bounded by

$$C_2(1+\lambda^\gamma)\left(1 + |x_0|^\gamma + \lambda^\gamma \int_0^t \mathbb{E}\left[\sup_{\rho\in[0,s]} |Z_\rho|^\gamma\right] ds\right).$$

Since $\lambda \in]0,1]$ we obtain

$$\mathbb{E}\left[\sup_{\rho\in[0,t]} |Z_\rho|^\gamma\right] \leq C_3 \left(1 + |x_0|^\gamma + \int_0^t \mathbb{E}\left[\sup_{\rho\in[0,s]} |Z_\rho|^\gamma\right] ds\right),$$

Consequently, using Gronwall's lemma, the result follows. □

Now, in order to perform (7.8), we make use of Malliavin calculus for deriving expression (7.9). Omitting λ in the notation Z_t^λ, we get

$$D_r Z_t = \sigma(r\lambda^2, \lambda Z_r + x_0)\mathbb{1}_{[r,1]}(t) + \lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0) D_r Z_s dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0) D_r Z_s ds.$$

Consequently

$$D_r Z_t = \sigma(r\lambda^2, \lambda Z_r + x_0)\mathcal{E}\left(\lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0) dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0) ds\right),\ r < t,$$

where $\mathcal{E}(S)$ denotes the Doléans exponential of the continuous semi-martingale

$$S_t = \lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0) dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0) ds,\ t \in [0,1].$$

We recall that, $\mathcal{E}_t(S) = \exp(S_t - \frac{1}{2}[S]_t)$. Consequently, for fixed $t \in]0,1]$, we have

$$< DZ_t, DZ_t >= \int_0^t \sigma^2(r\lambda^2, \lambda Z_r + x_0)\mathcal{E}^2(\lambda; r,t) dr, \tag{7.11}$$

where

$$\mathcal{E}(\lambda;r,t) = \exp\left(\lambda\int_r^t \sigma'(s\lambda^2,\lambda Z_s+x_0)dW_s + \lambda^2\int_r^t (b' - \frac{(\sigma')^2}{2})(s\lambda^2,\lambda Z_s+x_0)ds\right). \quad (7.12)$$

We set, for every $s \le t$, $G(\lambda,s) = \frac{D_s Z_t}{< DZ_t, DZ_t >}$. In view of the application of [27, Proposition 2.1.1], which implies the useful expression (7.25) for the density of Z_t, we will need to show that $G(\lambda,\cdot)$ belongs to the domain of the divergence operator δ, denoted by $Dom\ \delta$. It will be the case if $G(\lambda,\cdot) \in \mathbb{L}^{1,2}(H)$ with $H = L^2([0,T])$. In fact, by the lines after [27, Definition 1.3.2], we know that $\mathbb{L}^{1,2} \subset Dom\ \delta$. Since $Z_t \in \mathbb{D}^\infty$, we can deduce that $\frac{1}{< DZ_t, DZ_t >}$ belongs to $\mathbb{D}^\infty$, provided that we prove

$$\frac{1}{< DZ_t, DZ_t >} \in L^p(\Omega), \quad \forall p \ge 1, \quad (7.13)$$

see [27, Lemma 2.1.6]. Since $\mathbb{D}^\infty$ is an algebra, $G(\lambda,s) \in \mathbb{D}^\infty$, for $s \in]0,T]$, and so $G(\lambda,s) \in \mathbb{D}^{1,2}$. (7.13) will be the object of Proposition 7.5. According to [27, Definition 1.3.2], to affirm that $G(\lambda,\cdot)$ belongs to $\mathbb{L}^{1,2}$ it remains to show the existence of a measurable version of $D_{s_1}G(\lambda,s)$, $(s_1,s) \in [0,t]^2$, such that

$$\mathbb{E}\left[\int_{[0,t]^2} (D_{s_1}G(\lambda,s))^2 ds_1 ds\right] < +\infty. \quad (7.14)$$

We first state the following Lemma.

Lemma 7.4. *For every $q > 1$, there exists a constant $C_0(q)$ such that*

$$\sup_{0<r\le t\le 1,\lambda\in]0,1]} \mathbb{E}\left[(\mathcal{E}(\lambda;r,t))^q\right] \le C_0(q). \quad (7.15)$$

Proof of Lemma 7.4. Let $\lambda \in]0,1]$ and $q > 1$. For fixed $0 < r \le t \le 1$, (7.12) gives

$$\mathcal{E}^q(\lambda;r,t) = M(\lambda;r,t,q)\exp\left(\frac{\lambda^2(q^2-q)}{2}\int_r^t (\sigma')^2(\rho\lambda^2,\lambda Z_\rho+x_0)d\rho + q\lambda^2\int_r^t b'(\rho\lambda^2,\lambda Z_\rho+x_0)d\rho\right),$$

where

$$M(\lambda;r,t,q) = \exp\left(\int_r^t \lambda q\sigma'(\rho\lambda^2,\lambda Z_\rho+x_0)dW_\rho - \frac{1}{2}\int_r^t (q\lambda\sigma')^2(\rho\lambda^2,\lambda Z_\rho+x_0)d\rho\right). \quad (7.16)$$

In fact, since σ' is bounded, the stochastic exponential $M(\lambda;r,t,q)$ verifies Novikov's condition; therefore it is a martingale. So $\mathbb{E}(M(\lambda;r,t,q)) = 1$. In addition, since b' is also bounded and $\lambda \in]0,1]$, we get $\mathbb{E}\left[(\mathcal{E}(\lambda;r,t))^q\right] \le C_0(q)$, where $C_0(q) = \exp\left(2(q^2-q)\|\sigma'\|_\infty^2 + 2q\|b'\|_\infty\right)$. Consequently (7.15) is established. □

Proposition 7.5. *There is a constant $\mathcal{C}$ (not depending on x_0) such that*

$$\sup_{(t,\lambda)\in]0,1]^2} \mathbb{E}[(< DZ_t, DZ_t >)^{-p}] \le \mathcal{C}, \quad \forall p \ge 1.$$

Proof of Proposition 7.5. Let $t \in]0,1]$ fixed, $\epsilon_0 = \frac{c_0 t}{8}$, where c_0 is a non-degeneracy constant of σ^2 in the sense of Definition 2.1. Consider $\epsilon \in]0,\epsilon_0[$, we set $N := N^\lambda =<$

$DZ_t^\lambda, DZ_t^\lambda >$, where we recall that $< DZ_t, DZ_t >$ appears in (7.11) and (7.12). According to Proposition 7.1 we have to evaluate $P(N \leq \epsilon)$. Taking into account the lower bound of σ we have

$$\begin{aligned} P(N \leq \epsilon) &\leq P\left(\int_0^t dr \mathcal{E}^2(\lambda; r, t) \leq \frac{\epsilon}{c_0}\right) \\ &\leq P\left(\left(\int_{t-\frac{4\epsilon}{c_0}}^t dr \mathcal{E}^2(\lambda; r, t)\right)^{\frac{1}{2}} \leq \sqrt{\frac{\epsilon}{c_0}}\right) \\ &\leq P\left(\left(\int_{t-\frac{4\epsilon}{c_0}}^t dr\right)^{\frac{1}{2}} - \left(\int_{t-\frac{4\epsilon}{c_0}}^t \mathcal{E}^2(\lambda; r, t) dr\right)^{\frac{1}{2}} \geq \sqrt{\frac{\epsilon}{c_0}}\right). \end{aligned} \tag{7.17}$$

By the inverse triangle inequality of the $L^2([t - \frac{4\epsilon}{c_0}, t])$-norm we get

$$P(N \leq \epsilon) \leq P\left(\int_{t-\frac{4\epsilon}{c_0}}^t (1 - \mathcal{E}(\lambda; r, t))^2 dr \geq \frac{\epsilon}{c_0}\right).$$

Let $p \geq 1$. By Chebyshev's inequality this is lower than

$$\left(\frac{c_0}{\epsilon}\right)^{p+1} \mathbb{E}\left[\left(\int_{t-\frac{4\epsilon}{c_0}}^t (1 - \mathcal{E}(\lambda; r, t))^2 dr\right)^{p+1}\right].$$

Then, using Jensen's inequality, we get

$$P(N \leq \epsilon) \leq 4^p \left(\frac{\epsilon}{c_0}\right)^{-1} \int_{t-\frac{4\epsilon}{c_0}}^t \mathbb{E}\left[(1 - \mathcal{E}(\lambda; r, t))^{2(p+1)}\right] dr. \tag{7.18}$$

Furthermore (7.12) implies that $\mathcal{E}(\lambda; r, t)$ solves

$$\mathcal{E}(\lambda; r, t) = 1 + \lambda \int_r^t \mathcal{E}(\lambda; r, s) \sigma'(s\lambda^2, \lambda Z_s + x_0) dW_s + \lambda^2 \int_r^t \mathcal{E}(\lambda; r, s) b'(s\lambda^2, \lambda Z_s + x_0) ds.$$

Thus

$$\begin{aligned} \mathbb{E}\left[(\mathcal{E}(\lambda; r, t) - 1)^{2(p+1)}\right] &\leq 2^{2(p+1)} \mathbb{E}\left[\left|\lambda \int_r^t \mathcal{E}(\lambda; r, s) \sigma'(s\lambda^2, \lambda Z_s + x_0) dW_s\right|^{2(p+1)}\right] \\ &\quad + 2^{2(p+1)} \mathbb{E}\left[\left|\lambda^2 \int_r^t \mathcal{E}(\lambda; r, s) b'(s\lambda^2, \lambda Z_s + x_0) ds\right|^{2(p+1)}\right]. \end{aligned} \tag{7.19}$$

On one hand, using Jensen's inequality and $\lambda \in]0, 1]$, we obtain

$$\mathbb{E}\left[\left|\lambda^2 \int_r^t \mathcal{E}(\lambda; r, s) b'(s\lambda^2, \lambda Z_s + x_0) ds\right|^{2(p+1)}\right] \leq \|b'\|_\infty |t - r|^{2p+1} \int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda; r, s))^{2(p+1)}\right] ds. \tag{7.20}$$

On the other hand, by Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\left[\left|\lambda \int_r^t \mathcal{E}(\lambda; r, s) \sigma'(s\lambda^2, \lambda Z_s + x_0) dW_s\right|^{2(p+1)}\right] \leq \mathbb{E}\left[\left|\lambda^2 \int_r^t \mathcal{E}^2(\lambda; r, s) (\sigma')^2 (s\lambda^2, \lambda Z_s + x_0) ds\right|^{\mathrm{p+1}}\right].$$

Applying again Jensen's inequality gives

$$\mathbb{E}\left[\left|\lambda\int_r^t \mathcal{E}(\lambda;r,s)\sigma'(s\lambda^2,\lambda Z_s+x_0)dW_s\right|^{2(p+1)}\right]\leq \|\sigma'\|_\infty|t-r|^p\int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda;r,s))^{2(p+1)}\right]ds. \tag{7.21}$$

Therefore (7.20), (7.21) and (7.19) lead to

$$\mathbb{E}\left[(\mathcal{E}(\lambda;r,t)-1)^{2(p+1)}\right]\leq C(T,\|\sigma'\|_\infty,\|b'\|_\infty)|t-r|^p\int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda;r,s))^{2(p+1)}\right]ds. \tag{7.22}$$

By Lemma 7.4 and (7.22), there is a constant $C_0(2(p+1))$ such that

$$\mathbb{E}\left[(\mathcal{E}(\lambda;r,t)-1)^{2(p+1)}\right]\leq C_1(T,\|\sigma'\|_\infty,\|b'\|_\infty)C_0(2(p+1)). \tag{7.23}$$

Then, coming back to (7.18) and using (7.23), we obtain

$$\forall\epsilon\in]0,\epsilon_0],\quad P(N\leq\epsilon)\leq C(p)\epsilon^{p+1}, \tag{7.24}$$

where $C(p)=\dfrac{4^{2(p+1)}C_0(2(p+1))C_1(T,\|\sigma'\|_\infty,\|b'\|_\infty)}{p+1}$. Finally, using Proposition 7.1, the result follows. □

We go on with the proof of Proposition 5.1 taking into account the considerations before Lemma 7.4. In fact [27, Proposition 2.1.1] allows us to express, for fixed $t\in]0,1]$,

$$q_t^\lambda(z)=\mathbb{E}\left[\mathbb{1}_{\{Z_t>z\}}\delta(G(\lambda,\cdot))\right]; \tag{7.25}$$

using Cauchy-Schwarz inequality, it implies that

$$q_t^\lambda(z)\leq\sqrt{\mathbb{E}\left[|\delta(G(\lambda,\cdot))|^2\right]}. \tag{7.26}$$

According to (1.48) in [27], (7.26) implies

$$q_t^\lambda(z)\leq\left(\mathbb{E}\left[\int_0^t G^2(\lambda,s)ds\right]+\mathbb{E}\left[\int_{[0,t]^2}(D_{s_1}G(\lambda,s))^2ds_1ds\right]\right)^{\frac{1}{2}}. \tag{7.27}$$

Now we state a result that estimates the two terms in the right-hand side of (7.27). Indeed, we have the following.

Proposition 7.6. *For every $\lambda\in]0,1]$, $G(\lambda,\cdot)\in\mathbb{L}^{1,2}$. Moreover, the following statements hold: (i)* $\mathbb{E}\left[\int_0^t G^2(\lambda,s)ds\right]\leq\mathcal{C}_1\left(1+|x_0|^2\right)$, *(ii)* $\mathbb{E}\left[\int_{[0,t]^2}(D_{s_1}G(\lambda,s))^2ds_1ds\right]\leq\mathcal{C}_2\left(1+|x_0|^8\right)$, *where $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on T, $\|\sigma'\|_\infty$ and $\|b'\|_\infty$, but not on x_0.*

Proof of Proposition 7.6. (i) First we set $I_1=\mathbb{E}\left[\int_0^t G^2(\lambda,s)ds\right]$, recalling that

$$G(\lambda,s)=\frac{\sigma(s\lambda^2,\lambda Z_s+x_0)\mathcal{E}(\lambda;s,t)}{G_{den}},\quad\text{where }\ G_{den}=<DZ_t,DZ_t>.$$

By Cauchy-Schwarz inequality, we have

$$I_1\leq\left(\mathbb{E}\left[\int_0^t\sigma^4(s\lambda^2,\lambda Z_s+x_0)ds\right]\mathbb{E}\left[\int_0^t\frac{\mathcal{E}^4(\lambda;s,t)}{G_{den}^4}ds\right]\right)^{\frac{1}{2}}. \tag{7.28}$$

Since σ has linear growth, by Lemma 7.2 and using again Cauchy-Schwarz inequality, there is a constant C_1 such that

$$I_1 \leq C_1 \left(1+|x_0|^2\right) \left(\mathbb{E}\left[{G_{den}}^{-8}\right] \int_0^t \mathbb{E}\left[\mathcal{E}^8(\lambda; s,t)\right] ds \right)^{\frac{1}{4}}. \tag{7.29}$$

Consequently, using Proposition 7.5 and Lemma 7.4, the first item of Proposition 7.6 is established.

(ii) We set $I_2 = \mathbb{E}\left[\int_{[0,t]^2} (D_{s_1} G(\lambda,s))^2 ds_1 ds\right]$.

On one hand, by usual Malliavin differentiation rules, we obtain

$$\begin{aligned} D_{s_1} G(\lambda,s) = \quad & \lambda\sigma'(s\lambda^2, \lambda Z_s + x_0)\sigma(s_1\lambda^2, \lambda Z_{s_1}+x_0)\mathcal{E}(\lambda; s_1, s)\frac{\mathcal{E}(\lambda;s,t)}{G_{den}}\mathbb{1}_{[0,s]}(s_1) \\ + \quad & \sigma(s\lambda^2, \lambda Z_s + x_0)\frac{D_{s_1}\mathcal{E}(\lambda;s,t)}{G_{den}} - \sigma(s\lambda^2, \lambda Z_s + x_0)\mathcal{E}(\lambda;s,t)\frac{D_{s_1} G_{den}}{G_{den}^2}. \end{aligned} \tag{7.30}$$

The right-hand side being measurable with respect to $\Omega \times [0,T]^2$, $G(\lambda,\cdot)$ will belong to $\mathbb{L}^{1,2}$ if (ii) is established. At this point we need to evaluate $D_{s_1}\mathcal{E}(\lambda;s,t)$. From now on, for the sake of simplicity, we will only expose the calculations in the case when $b \equiv 0$. In fact, we have

$$\begin{aligned} D_{s_1}\mathcal{E}(\lambda;s,t) &= \mathcal{E}(\lambda;s,t) D_{s_1}\left(\lambda \int_s^t \sigma'(\ell\lambda^2, \lambda Z_\ell + x_0) dW_\ell - \frac{\lambda^2}{2}\int_s^t (\sigma')^2(\ell\lambda^2, \lambda Z_\ell + x_0) d\ell\right) \\ &= \mathcal{E}(\lambda;s,t)\Bigg(\lambda\sigma'(s_1\lambda^2, \lambda Z_{s_1}+x_0)\mathbb{1}_{[s,t]}(s_1) \\ &+ \lambda^2\sigma(s_1\lambda^2, \lambda Z_{s_1}+x_0)\int_s^t \mathbb{1}_{[s_1,t]}(\ell)\sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda;s_1,\ell) dW_\ell \\ &- \lambda^3\sigma(s_1\lambda^2, \lambda Z_{s_1}+x_0)\int_s^t \mathbb{1}_{[s_1,t]}(\ell)(\sigma'\sigma'')(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda;s_1,\ell) d\ell\Bigg). \end{aligned} \tag{7.31}$$

On the other hand, we get

$$\begin{aligned} D_{s_1} G_{den} = 2\lambda &\int_0^t \sigma\sigma'(\xi\lambda^2, \lambda Z_\xi + x_0)\mathbb{1}_{[s_1,t]}(\xi)\sigma(s_1\lambda^2, \lambda Z_{s_1}+x_0)\mathcal{E}(\lambda;s_1,\xi)\mathcal{E}^2(\lambda;\xi,t) d\xi \\ + 2&\int_0^t \sigma^2(\xi\lambda^2, \lambda Z_\xi + x_0)\mathcal{E}(\lambda;\xi,t) D_{s_1}\mathcal{E}(\lambda;\xi,t) d\xi. \end{aligned} \tag{7.32}$$

Therefore, coming back to (7.30) and using (7.31), we obtain that

$$I_2 \leq 4\left[J_1 + J_2 + J_3\right], \tag{7.33}$$

where

$$J_1 = \mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \left|\sigma'(s\lambda^2, \lambda Z_s + x_0)\sigma(s_1\lambda^2, \lambda Z_{s_1} + x_0)\mathcal{E}(\lambda; s_1, s)\frac{\mathcal{E}(\lambda; s, t)}{G_{den}}\right|^2\right],$$

$$J_2 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 ds \sigma^2(s\lambda^2, \lambda Z_s + x_0)\frac{\mathcal{E}^2(\lambda; s, t)}{G_{den}^2}\left(\mathbb{1}_{[s,t]}(s_1)\sigma'(s_1\lambda^2, \lambda Z_{s_1} + x_0)\right.\right.$$

$$\left.\left. + \ \sigma(s_1\lambda^2, \lambda Z_{s_1} + x_0)\int_s^t \mathbb{1}_{[s_1,t]}(\ell)\sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda; s_1, \ell)\left[dW_\ell - \sigma'\sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)d\ell\right]\right)^2\right],$$

$$J_3 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 ds \frac{\sigma^2(s\lambda^2, \lambda Z_s + x_0)}{G_{den}^4}\mathcal{E}^2(\lambda; s, t)\left(D_{s_1}G_{den}\right)^2\right],$$

with $D_{s_1}G_{den}$ given in (7.32). In the sequel we will enumerate constants K_1 to K_{20}; all those will not depend on x_0 or t, but eventually on T, σ and b. We start estimating J_1. Since σ' is bounded, by Cauchy-Schwarz inequality, we have

$$J_1 \leq K_1\left(\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \sigma^4(s_1\lambda^2, \lambda Z_{s_1} + x_0)\right]\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^4(\lambda; s, t)\frac{\mathcal{E}^4(\lambda; s_1, s)}{G_{den}^4}\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, Lemma 7.2 and a further use of Cauchy-Schwarz inequality imply that, J_1 is bounded by

$$K_2\left(1 + |x_0|^2\right)\left(\mathbb{E}\left[G_{den}^{-8}\right]\right)^{\frac{1}{4}}\left(\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^{16}(\lambda; s, t)\right]\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^{16}(\lambda; s_1, s)\right]\right)^{\frac{1}{8}}.$$

Therefore, by Proposition 7.5 and Lemma 7.4, we obtain that

$$J_1 \leq K_3\left(1 + |x_0|^2\right). \tag{7.34}$$

We go on with the analysis of J_2. Since σ', σ'' are bounded, we have

$$J_2 \leq K_4\mathbb{E}\left[\int_{[0,t]^2} ds ds_1 \sigma^2(s\lambda^2, \lambda Z_s + x_0)\frac{\mathcal{E}^2(\lambda; s, t)}{G_{den}^2}\left(1 + \sigma^2(s_1\lambda^2, \lambda Z_{s_1} + x_0)M^2(s, s_1; t)\right)\right],$$

where $M(s, s_1; t) = \int_{s\vee s_1}^t \sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda; s_1, \ell)dW_\ell$, $t\geq s\vee s_1$, is a martingale having all moments because of Lemma 7.4. Since σ has linear growth, using Cauchy-Schwarz inequality and Lemma 7.2, we get

$$J_2 \leq K_5(1 + |x_0|^4)\left(\mathbb{E}\left[\int_0^t ds \mathcal{E}^{16}(\lambda; s, t)\right]\mathbb{E}\left[G_{den}^{-16}\right]\right)^{\frac{1}{8}}\left(\int_{[0,t]^2} ds ds_1 E[M^8(s, s_1; t)]\right)^{\frac{1}{4}}.$$

Then Burkholder-Davis-Gundy inequality, Lemma 7.4 and Proposition 7.5 imply

$$J_2 \leq K_6(1 + |x_0|^4). \tag{7.35}$$

Finally we treat J_3. Applying Cauchy-Schwarz inequality, we have

$$J_3 \leq \left(\mathbb{E}\left[\int\limits_{[0,t]^2} ds ds_1 \frac{\sigma^4(s\lambda^2, \lambda Z_s + x_0)}{G_{den}^8} \mathcal{E}^4(\lambda; s, t) \right] \mathbb{E}\left[\int\limits_{[0,t]^2} ds ds_1 (D_{s_1} G_{den})^4 \right] \right)^{\frac{1}{2}}.$$

Since σ has linear growth, again by Cauchy-Schwarz inequality and Lemma 7.2, we get

$$J_3 \leq K_7(1 + |x_0|^2) \left(\mathbb{E}\left[G_{den}^{-32} \right] \int\limits_0^t ds \mathbb{E}\left[\mathcal{E}^{16}(\lambda; s, t) \right] \right)^{\frac{1}{8}} \left(\mathbb{E}\left[\int\limits_0^t ds_1 (D_{s_1} G_{den})^4 \right] \right)^{\frac{1}{2}};$$

by Lemma 7.4 and Proposition 7.5 it follows

$$J_3 \leq K_8(1 + |x_0|^2) \left(\mathbb{E}\left[\int\limits_0^t ds_1 (D_{s_1} G_{den})^4 \right] \right)^{\frac{1}{2}}. \tag{7.36}$$

Since σ' is bounded, (7.32) and Jensen's inequality give

$$\mathbb{E}\left[\int\limits_0^t ds_1 (D_{s_1} G_{den})^4 \right] \leq K_9 \left(A_1 + A_2 \right), \tag{7.37}$$

where

$$A_1 = \mathbb{E}\left[\int\limits_0^t ds_1 \int\limits_{s_1}^t d\xi \sigma^4(s_1\lambda^2, \lambda Z_{s_1} + x_0) \sigma^4(\xi\lambda^2, \lambda Z_\xi + x_0) \mathcal{E}^4(\lambda; s_1, \xi) \mathcal{E}^8(\lambda; \xi, t) \right],$$

$$A_2 = \mathbb{E}\left[\int\limits_0^t ds_1 \int\limits_0^t d\xi \sigma^8(\xi\lambda^2, \lambda Z_\xi + x_0) \mathcal{E}^4(\lambda; \xi, t) \left(D_{s_1} \mathcal{E}(\lambda; \xi, t) \right)^4 \right].$$

Since σ has linear growth, Cauchy-Schwarz inequality and Lemma 7.2 imply that A_1 is bounded by

$$K_{10} \left(1 + |x_0|^8 \right) \left(\mathbb{E}\left[\int\limits_0^t ds_1 \int\limits_{s_1}^t d\xi \mathcal{E}^8(\lambda; s_1, \xi) \mathcal{E}^{16}(\lambda; \xi, t) \right] \right)^{\frac{1}{2}}.$$

Again, by Cauchy-Schwarz inequality and Lemma 7.4, we obtain

$$A_1 \leq K_{11} \left(1 + |x_0|^8 \right). \tag{7.38}$$

We proceed estimating A_2. Using Cauchy-Schwarz inequality, A_2 is bounded by

$$K_{12} \left(\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \sigma^{16}(\xi\lambda^2, \lambda Z_\xi + x_0) \right] \mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \mathcal{E}^8(\lambda; \xi, t) \left(D_{s_1} \mathcal{E}(\lambda; \xi, t) \right)^8 \right] \right)^{\frac{1}{2}}.$$

Since σ has linear growth, Cauchy-Schwarz inequality and Lemma 7.2 lead to

$$A_2 \leq K_{13} \left(1 + |x_0|^8 \right) \left(\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \mathcal{E}^{16}(\lambda; \xi, t) \right] \mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \left(D_{s_1} \mathcal{E}(\lambda; \xi, t) \right)^{16} \right] \right)^{\frac{1}{4}};$$

Lemma 7.4 implies

$$A_2 \leq K_{14}\left(1+|x_0|^8\right)\left(\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^{16}\right]\right)^{\frac{1}{4}}. \tag{7.39}$$

Since σ' and σ'' are bounded, using (7.31) and Jensen's inequality, it follows that

$$\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^{16}\right] \leq K_{15}\left(R_1+R_2\right), \tag{7.40}$$

where

$$R_1 = \mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \mathcal{E}^{16}(\lambda;\xi,t)\right],$$

$$R_2 = \mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \sigma^{16}(s_1\lambda^2, \lambda Z_{s_1}+x_0)\mathcal{E}^{16}(\lambda;\xi,t)\left(M^{16}(s_1,\xi;t)+\int\limits_{\xi\vee s_1}^{t}\mathcal{E}^{16}(\lambda;s_1,\rho)d\rho\right)\right],$$

and $M(s_1,\xi;t) = \int\limits_{\xi\vee s_1}^{t} \sigma''(\rho\lambda^2, \lambda Z_\rho + x_0)\mathcal{E}(\lambda;s_1,\rho)dW_\rho$, $t\geq \xi\vee s_1$, is a square integrable martingale, taking into account Lemma 7.4. Again by Lemma 7.4, R_1 is uniformly bounded in t and x_0. On the other hand, using Cauchy-Schwarz and Jensen's inequalities, R_2 is bounded by

$$K_{16}\left(\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \sigma^{32}(s_1\lambda^2, \lambda Z_{s_1}+x_0)\mathcal{E}^{32}(\lambda;\xi,t)\right]\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi \left(M^{32}(s_1,\xi;t)+\int\limits_{\xi\vee s_1}^{t}\mathcal{E}^{32}(\lambda;s_1,\rho)d\rho\right)\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, again by Cauchy-Schwarz inequality, Lemma 7.2 and Lemma 7.4, we get

$$R_2 \leq K_{17}\left(1+|x_0|^{16}\right)\mathbb{E}\left[\int\limits_{[0,t]^2} ds_1 d\xi M^{32}(s_1,\xi;t)\right].$$

Burkholder-Davis-Gundy inequality gives

$$R_2 \leq K_{18}\left(1+|x_0|^{16}\right). \tag{7.41}$$

Coming back to (7.39), using (7.41) and (7.40), we obtain

$$A_2 \leq K_{19}\left(1+|x_0|^{12}\right); \tag{7.42}$$

thus, replacing (7.38) and (7.42) in (7.37) and coming back to (7.36), imply

$$J_3 \leq K_{20}\left(1+|x_0|^{8}\right). \tag{7.43}$$

Consequently, substituting (7.34), (7.35) and (7.43) in (7.33), item (ii) of Proposition 7.6 is established. □

Returning to the proof of Proposition 5.1 and substituting in (7.27) the right-hand side of the first and the second item of Proposition 7.6, the inequality (5.2) is verified. Finally this concludes the proof of Proposition 5.1. □

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